

# On the Borel-Cantelli lemma

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In the present note, we generalize the first part of the Borel-Cantelli lemma. By this generalization, we obtain some strong limit results.

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## 1 Introduction

Suppose  $A_1, A_2, \dots$  is a sequence of events on a common probability space and that  $A_i^c$  denotes the complement of event  $A_i$ . The Borel-Cantelli lemma, presented below as Lemma 1.1, is used extensively for producing strong limit theorems.

**Lemma 1.1.** 1. *If, for any sequence  $A_1, A_2, \dots$  of events,*

$$\sum_{n=1}^{\infty} P(A_n) < \infty, \quad (1.1)$$

*then  $P(A_n \text{ i.o.}) = 0$ , where i.o. is an abbreviation for "infinitively often".*

2. *If  $A_1, A_2, \dots$  is a sequence of independent events and if*

$$\sum_{n=1}^{\infty} P(A_n) = \infty, \quad (1.2)$$

*then  $P(A_n \text{ i.o.}) = 1$ .*

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The independence condition in the second part of the Borel-Cantelli lemma is weakened by a number of authors, including Chung and Erdos (1952), Erdos and Renyi (1959), Lamperti (1963), Kochen and Stone (1964), Spitzer (1964), Ortega and Wschebor (1983), and Petrov (2002), (2004). One can also refer to Martikainen and Petrov (1990), and Petrov (1995) for related topics.

The first part of the Borel-Cantelli lemma is generalized in Barndorff-Nielsen (1961) (Lemma 1.2) and Balakrishnan and Stepanov (2010) (Lemma 1.3).

**Lemma 1.2.** *Let  $A_1, A_2, \dots$  be a sequence of events such that  $P(A_n) \rightarrow 0$ . If*

$$\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty, \quad (1.3)$$

*then  $P(A_n \text{ i.o.}) = 0$ .*

**Lemma 1.3.** *Let  $A_1, A_2, \dots$  be a sequence of events such that  $P(A_n) \rightarrow 0$ . If*

$$\sum_{n=1}^{\infty} P(A_n^c A_{n+1}) < \infty, \quad (1.4)$$

*then  $P\{A_n \text{ i.o.}\} = 0$ .*

In the present work, by simple rewriting conditions in (1.3) and (1.4) we obtain a new generalization of the first part of the Borel-Cantelli lemma. We will show that  $P(A_n \text{ i.o.}) = 0$  might be also true when  $P(A_n) \rightarrow 0$  and inequality (1.2) holds. This generalization is illustrated by some limit results.

The rest of this paper is organized as follows. In Section 2, we present our results. In Section 3, we use the results of Section 2 for deriving strong limit theorems for dependent random variables.

## 2 Results

**Lemma 2.1.** *Let  $A_1, A_2, \dots$  be a sequence of events such that  $P(A_n) \rightarrow 0$ . Let (1.2) hold true,*

$$\sum_{n=1}^{\infty} P(A_n A_{n+1}) = \infty \quad (2.1)$$

*and*

$$\sum_{n=1}^{\infty} [P(A_n) - P(A_n A_{n+1})] < \infty. \quad (2.2)$$

*Then  $P(A_n \text{ i.o.}) = 0$ .*

**Proof of Lemma 2.1** Observe that  $P(A_n A_{n+1}^c) = P(A_n) - P(A_n A_{n+1})$ . The truth of Lemma 2.1 follows from Lemma 1.2.  $\square$

**Remark 2.1.** Condition (2.2) in Lemma 2.1 might be replaced by the condition

$$\sum_{n=1}^{\infty} [P(A_{n+1}) - P(A_n A_{n+1})] < \infty.$$

In that case we could prove Lemma 2.1 by using Lemma 1.3 instead of Lemma 1.2.

### 3 Applications: limit results

The importance of the theoretical results of Section 2 is shown in this section. The following limit theorem can be easily derived from Lemma 2.1.

**Theorem 3.1.** Let  $X_1, X_2, \dots$  be a sequence of dependent random variables such that  $X_n \xrightarrow{p} \mu$ , where  $\mu$  is a constant. Let for all small  $\varepsilon > 0$ ,

$$\sum_{n=1}^{\infty} P(X_n \notin [\mu - \varepsilon, \mu + \varepsilon]) = \infty,$$

$$\sum_{n=1}^{\infty} P(X_n \notin [\mu - \varepsilon, \mu + \varepsilon], X_{n+1} \notin [\mu - \varepsilon, \mu + \varepsilon]) = \infty,$$

and

$$\sum_{n=1}^{\infty} [P(X_n \notin [\mu - \varepsilon, \mu + \varepsilon]) - P(X_n \notin [\mu - \varepsilon, \mu + \varepsilon], X_{n+1} \notin [\mu - \varepsilon, \mu + \varepsilon])] < \infty.$$

Then

$$X_n \xrightarrow{a.s.} \mu.$$

Theorem 3.1 is illustrated by Example 3.1 below. Here, we also formulate two trivial statements: Proposition 3.1 and Corollary 3.1. These statements are given to simplify the presentation of Example 3.1.

**Proposition 3.1.** Let  $A_1, A_2, \dots$  be a decreasing sequence of events such that  $P(A_n) \rightarrow 0$ . Then  $P(A_n \text{ i.o.}) = 0$ .

Proposition 3.1 can be easily proved directly. Corollary 3.1 follows from Proposition 3.1.

**Corollary 3.1.** Let  $X_1, X_2, \dots$  be an ordered sequence of random variables such that  $X_n \xrightarrow{p} \mu$ , where  $\mu$  is a constant. Then  $X_n \xrightarrow{a.s.} \mu$ .

**Example 3.1.** Let  $X_1, \dots, X_n, \dots$  be a sequence of dependent random variables defined for any  $n \geq 1$  by the Clayton copula

$$F(x_1, x_2, \dots, x_n) = \left[ \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} - (n-1) \right]^{-1},$$

where  $0 < x_i < 1$  ( $1 \leq i \leq n$ ). Let  $M_n = \max\{X_1, \dots, X_n\}$ . Then, for any  $x \in (0, 1)$  and  $n \rightarrow \infty$

$$P(M_n \leq x) = \left[ n \left( \frac{1}{x} - 1 \right) + 1 \right]^{-1} \rightarrow 0.$$

We see that  $M_n$  converges in probability to 1. Since  $\sum_{n=1}^{\infty} P(M_n \leq x) = \infty$ , we can not apply the first part of the Borel-Cantelli lemma for deriving the corresponding strong limit result. However,  $M_n$  is an ordered sequence of random variables. By Corollary 3.1, we have

$$M_n \xrightarrow{a.s.} 1.$$

Theorem 3.1 allows us to obtain a more elaborate strong limit result. We will show that

$$M_n^{\alpha} \xrightarrow{a.s.} 1 \quad (0 < \alpha < 1). \quad (3.1)$$

Observe first that

$$P(M_n^{\alpha} \leq x) = \left[ n \left( x^{-n^{-\alpha}} - 1 \right) + 1 \right]^{-1} \sim (-\log x) n^{\alpha-1} \rightarrow 0.$$

It follows that  $M_n^{\alpha} \xrightarrow{p} 1$ . However, we can not apply here Proposition 3.1, since  $M_n^{\alpha}$  is not an ordered sequence. Instead, we utilize Theorem 3.1. Observe that

$$\sum_{n=1}^{\infty} P(M_n^{\alpha} \leq x) = \infty,$$

$$\sum_{n=1}^{\infty} P(M_n^{\alpha} \leq x, M_{n+1}^{(n+1)\alpha} \leq x) = \sum_{n=1}^{\infty} \frac{1}{n(x^{-n^{-\alpha}} - 1) + x^{-(n+1)^{-\alpha}}} = \infty$$

and

$$\begin{aligned} & \sum_{n=1}^{\infty} [P(M_n^{\alpha} \leq x) - P(M_n^{\alpha} \leq x, M_{n+1}^{(n+1)\alpha} \leq x)] = \\ & \sum_{n=1}^{\infty} \frac{x^{-(n+1)^{-\alpha}} - 1}{(n(x^{-n^{-\alpha}} - 1) + 1)(n(x^{-n^{-\alpha}} - 1) + x^{-(n+1)^{-\alpha}})} \sim \\ & \frac{1}{(-\log x)} \sum_{n=1}^{\infty} \frac{1}{n^{2-\alpha}} < \infty, \end{aligned}$$

The convergence in (3.1) readily follows.

## References

- Balakrishnan, N., Stepanov, A. (2010). Generalization of the Borel-Cantelli lemma. *The Mathematical Scientist*, **35** (1), 61–62.
- Barndorff-Nielsen, O. (1961). On the rate of growth of the partial maxima of a sequence of independent identically distributed random variables. *Math. Scand.*, **9**, 383–394.
- Chung, K.L. and Erdos, P. (1952). On the application of the Borel-Cantelli lemma. *Trans. Amer. Math. Soc.*, **72**, 179–186.
- Erdos, P. and Renyi, A. (1959). On Cantor’s series with convergent  $\sum 1/q_n$ . *Ann. Univ. Sci. Budapest. Sect. Math.*, **2**, 93–109.
- Kochen, S.B. and Stone, C.J. (1964). A note on the Borel-Cantelli lemma. *Illinois J. Math.*, **8**, 248–251.
- Lamperti, J. (1963). Wiener’s test and Markov chains. *J. Math. Anal. Appl.*, **6**, 58–66.
- Martikainen, A.I., Petrov, V.V., (1990). On the Borel-Cantelli lemma. *Zapiski Nauch. Semin. Leningrad. Otd. Steklov Mat. Inst.*, **184**, 200–207 (in Russian). English translation in: (1994). *J. Math. Sci.*, **63**, 540–544.
- Petrov, V.V. (1995). *Limit Theorems of Probability Theory*. Oxford University Press, Oxford.
- Petrov, V.V. (2002). A note on the Borel-Cantelli lemma. *Statist. Probab. Lett.*, **58**, 283–286.
- Petrov, V.V. (2004). A generalization of the Borel-Cantelli Lemma. *Statist. Probab. Lett.*, **67**, 233–239.
- Ortega, J., Wschebor, M., (1983). On the sequence of partial maxima of some random sequences. *Stochastic Process. Appl.*, **16**, 8598.
- Spitzer, F. (1964). *Principles of Random Walk*. Van Nostrand, Princeton, New Jersey.